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Precoloring extension for 2-connected graphs with maximum degree three

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ABSTRACT

Let $G = G(V, E)$ be a simple graph, $\mathcal{L}$ a list assignment with $|L(v)| = \Delta(G) \forall v \in V$ and $W \subseteq V$ an independent subset of the vertex set. Define $d(W) := \min\{d(v, w) \mid v, w \in W\}$ to be the minimum distance between two vertices of W. In this paper it is shown that if G is 2-connected with $\Delta(G) = 3$ and G is not K_4 then every precoloring of W is extendable to a proper list coloring of G provided that $d(W) \geq 6$. An example shows that the bound is sharp. This result completes the investigation of precoloring extensions for graphs with $|L(v)| = \Delta(G)$ for all $v \in V$ where the precolored set W is an independent set.

1. Introduction

Let us consider simple graphs $G = (V, E)$ with maximum degree $\Delta(G) \geq 3$. The well-known theorem of Brooks [10] states that such a graph is k-colorable if it does not contain K_{k+1} as a component. The aim of this paper is a generalization of this theorem.

First, we consider the list version of this problem. That means every vertex has a set $L(v)$ of available colors. The set $L(v)$ is also called a *list* of v and the collection of all lists is called a list assignment $\mathcal{L}$ of G. The graph G is $\mathcal{L}$*-list colorable* if a proper coloring of the vertices exists where every vertex gets a color from its list in $\mathcal{L}$. This concept was introduced by Erdős, Rubin and Taylor [11] and independently by Vizing [14]. A K*-assignment* is a list assignment $\mathcal{L}$ where $|L(v)| = k$ for all $v \in V$. Among others in [11,13] a Brooks-type theorem is proved saying that a graph G with maximum degree $k = \Delta(G) \geq 3$ is $\mathcal{L}$-list colorable for every k-assignment $\mathcal{L}$ if G does not contain K_{k+1}.

Let us define a *supervalent list assignment* $\mathcal{L}$ being a list assignment with $|L(v)| \geq d_G(v)$ for all $v \in V(G)$. Investigating the question whether a graph with supervalent list assignment is list colorable we need a special class of graphs. A *Gallai tree* is a connected graph in which every block is a complete graph or an odd cycle.

An important tool for the proof of the main result of this paper is the following theorem which is part of a more extensive result of [9,11,13], respectively.

Theorem 1 *([9,11,13]). If $\mathcal{L}$ is a supervalent list assignment for a connected graph G and there is no $\mathcal{L}$-coloring of G, then*

(a) $|L(v)| = d(v)$ *for every vertex* $v \in V(G)$.
(b) *G is a Gallai tree.*

Now we assume additionally that there is a subset $W \subseteq V$ of the vertex set which is already precolored. Denote by $d(W)$ the minimum distance between two components of W in G. We would like to extend the precoloring of W to a proper coloring of the whole vertex set. Clearly the existence of such an extension depends on $d(W)$ and the number of available colors or the length of the lists of the list assignment, respectively. First results were given by Albertson 1998 [1] dealing

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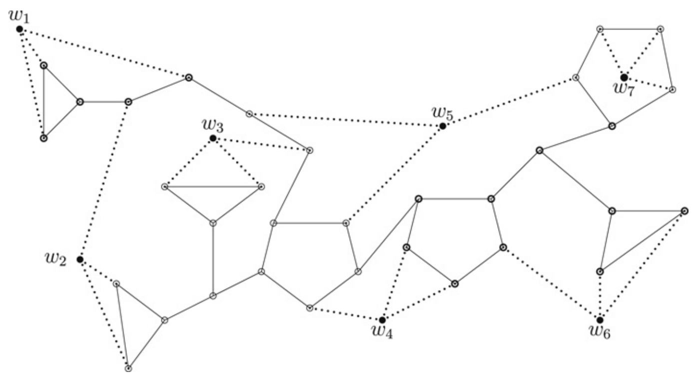

Fig. 1. Example with $d(W) = 3$.

with ordinary colorings. There were several papers in the past few years considering this topic from different points of view, see for example [2–8,12].

Here we ask for the extension of a precoloring of an independent set W to a proper list coloring if every vertex has a list of $\Delta(G) = 3$ colors. Axenovich [8] and Albertson, Kostochka and West [5] proved that for independent W, $k = \Delta(G) \geq 3$ and $d(W) \geq 8$ such an extension is always possible if G does not contain a K_{k+1} as subgraph. Furthermore they gave examples showing that the bound 8 is sharp. Remarkably, the mentioned examples are 1-connected graphs. In fact, for k-connected graphs G with $k \geq 2$ and $\Delta(G) \geq 4$ the requirement $d(W) \geq 4$ is already sufficient to guarantee an extension of such a precoloring to a proper list coloring. This result is proved in [15] where also examples were given showing the sharpness of this bound. Moreover in [15] it is pointed out that analogous results for $\Delta(G) = 2$ do not exist.

Thus it remains only one open case in this field. What happens if G is at least 2-connected and $\Delta(G) = 3$? In this paper this case is solved and an example is given showing that the bound is sharp.

2. Precoloring extension for $\Delta(G) = 3$

The main result of this paper is given in the following theorem.

Theorem 2. *Let $G = (V, E)$ be a 2-connected graph with $\Delta(G) = 3$ which is not K_4, $W \subseteq V$ an independent subset of the vertex set, $\mathcal{L}$ a list assignment with $|L(v)| = 3$ for all $v \in V$ and $d(W) \geq 6$. Then every precoloring of W extends to a proper $\mathcal{L}$-list coloring of V.*

Proof. Assume that the statement of the theorem is not true and G is a smallest counterexample.

Delete the colors of the precoloring of W from the lists of the corresponding neighbors. Denote the new list assignment by $\mathcal{L}'$ and the graph induced by $V(G) \setminus W$ by H. Because of $d(W) \geq 6$ we know that $|L'(v)| \geq 2,\ \forall v \in V(H)$. Note furthermore that $\mathcal{L}'$ is a supervalent list assignment for H since $|L(v)| = 3 \geq d_G(v)$ and therefore $|L'(v)| \geq d_H(v)$ for all $v \in V(H)$. By the minimality of G we may assume that H is connected.

Since G is a counterexample to the statement of the theorem it follows that H is not list colorable from the lists of $\mathcal{L}'$. Thus we have especially by Theorem 1

Claim 1. (a) *H is a Gallai tree.*
(b) $|L'(v)| = d_H(v)$
(c) *$d_G(v) = 3$ for all $v \in V(H)$.*

Denote the set of the non-cut vertices of a block B of H by B'. A *leaf block* of H is a block of H containing at most one cut vertex.

Note that we always have $d_H(v) = |L'(v)| \geq 2$. Thus all vertices of H have degree 2 or 3. Therefore each block of H is either an odd cycle or K_2 and each leaf block of H is an odd cycle. Fig. 1 shows an example for $d(W) = 3$. We will prove in the following that such a structure does not exist if $d(W) \geq 6$.

Claim 2. *Let B be a leaf block of H in a smallest counterexample G.*

(a) *It exists a unique vertex $w_B \in W$ which is adjacent to all non-cut vertices of B.*
(b) $B = K_3$.
(c) *H has more than one block.*
(d) *w_B has exactly one neighbor y_B in $V(H) \setminus \bigcup_{B \in \mathcal{B}_\ell} V(B)$ where $\mathcal{B}_\ell$ is the set of all leaf blocks of H.*

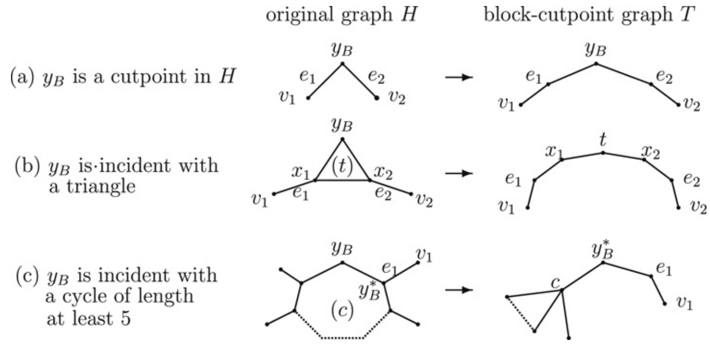

Fig. 2. Corresponding structures in H and T.

Proof of the Claim. (a) If H has only one block then H is an odd cycle. In this case we have $B = B' = H$.

The vertices of B' have degree 2 in H. Thus each of these vertices is adjacent to a vertex of W, otherwise we have $|L'(v)| > d_H(v)$, contradiction. Assume that the vertices of B' are adjacent to at least two different vertices of W. Since B' is connected in H (B is a block) it follows that there are adjacent vertices in B' which have different neighbors in W contradicting $d(W) \geq 6$.

(b) If B is a cycle of length at least 5 then because of (a) there has to be a vertex $w \in W$ which is adjacent to all – at least 4 – non-cut vertices. Thus $d_G(w) \geq 4$, contradiction. Consequently, $B = C_3 = K_3$.

(c) If H has only one block then because of (a) and (b) G has to be K_4 — contradiction.

(d) Because of (a) and (b) the vertex w_B has exactly two neighbors in B and B contains a cut vertex because of (c). If w_B has no further neighbor in H then the cut vertex of B in H would be a cut vertex in G too contradicting the 2-connectedness of G.

Thus w_B has exactly one neighbor in H outside B.

This neighbor cannot belong to another leaf block since then w_B has to be adjacent also to the second non-cut vertex of that leaf block because of (a). This contradicts $\Delta(G) = 3$. □

Let us define a new graph T associated with H. The *block-cutpoint graph* T of H has a vertex for each block in H and a vertex for each cut vertex of H and a cut vertex v is adjacent to a block B in T if $v \in V(B)$ in H. Note that the block-cutpoint graph of a connected graph H is a tree and every leaf of T corresponds to a leaf block of H.

Let B be a leaf block of H and y_B the neighbor of the corresponding w_B outside B (see Claim 2). Note that each y_B has degree 2 in H. Furthermore let T be the block-cutpoint graph of H. Let the notation of a vertex in the block-cutpoint graph T be the same as the notation of the corresponding element in the original graph H. In Fig. 2, the triangle of H is denoted by t in T and the cycle of H is denoted by c in T.

Now, we shall show that for every leaf block in H we can find a special structure in T. With the help of this structure we shall get a bound for the number of leaves of T which gives a contradiction to the number of leaf blocks in H.

First, for every leaf block B we would like to find a path in T such that two paths belonging to different leaf blocks have at most one end vertex in common.

If y_B (see Claim 2) is incident to an odd cycle of length at least 5 then denote one of the neighbors of y_B on the cycle by y_B^*. Note that y_B^* is a cutpoint of H incident to the cycle and a K_2.

Claim 3 (*See* Fig. 2).

1. *If y_B is a cutpoint in H then the corresponding vertex has degree* 2 *in T.*
2. *If y_B is incident to a triangle in H then the vertex corresponding to the triangle has degree* 2 *in T.*
3. *If y_B is incident to an odd cycle of length at least* 5 *in H then the vertex corresponding to y_B^* has degree* 2 *in T.*

So by Claim 3 we can assign a vertex $z_B \in V(T)$ with $d_T(z_B) = 2$ to each leaf block B belonging to H ($z_B \in \{y_B, t, y_B^*\}$).

Now we shall assign a path of the tree T to every leaf block B of H. For a given B the path contains z_B, all vertices except the end vertices have degree 2 in T and the end vertices have degree at least 3 in T.

We have to show that for every z_B such a path exists. Moreover we would like to prove that paths belonging to different leaf blocks in H do not have common inner vertices — every leaf block has its own path.

Claim 4. *For every z_B there exists such a path and there is no inner vertex v on the path such that $v = z_{B'}$ for a $B' \neq B$.*

Proof. We start at a fixed vertex z_B in T and follow the path in both directions. We have to show that in any case we will meet a vertex of degree at least 3 in T and there is no $z_{B'}$ between z_B and the vertex of degree at least 3.

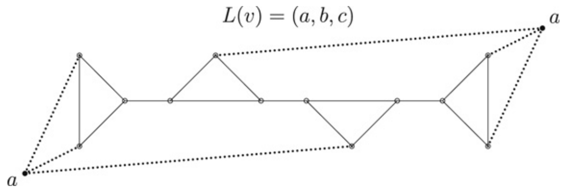

Fig. 3. Non-extendable precoloring for $\Delta(G) = 3$ and $d(W) = 5$.

1. Let us start with the case that y_B is incident to a 3-cycle, that is $z_B = t$ (Fig. 2 (b)). Denote the neighbors of y_B in H by x_1 and x_2. Both of them are cut vertices and the corresponding vertices in T have degree 2. Denote the third neighbor of x_i in H by v_i. Note that v_i has distance 2 to y_B in H and distance 3 to w_B in G. Therefore it cannot represent $z_{B'}$ because of $d(W) \geq 6$. Let us mention that in this case $d(W) \geq 5$ is not sufficient for this argument.
   We shall show that for $i \in \{1, 2\}$ either the vertex in T corresponding to v_i or a neighbor of this vertex in T (corresponding to a block in H) has degree at least 3 in T.
   If v_i is not incident to a cycle in H then it is incident to three K_2. Hence the corresponding vertex in T has degree 3.
   If v_i is incident to a 3-cycle in H then the 3-cycle cannot be incident to a further $y_{B'}$ because of $d(W) \geq 6$. Therefore the 3-cycle is incident to 3 cut vertices in H and the corresponding vertex in T has degree 3. Again, $d(W) \geq 5$ would not be sufficient for this argument (see Fig. 3).
   If v_i is incident to a cycle of length at least 5 then the degree of the vertex in T corresponding to the cycle in H is greater than 3.
2. Now assume that y_B is a cut vertex in H, thus $z_B = y_B$. Denote the neighbors of y_B in H by v_1 and v_2.
   For v_i in T we may argue as in the previous case.
3. Let y_B be incident to a cycle of length at least 5, $z_B = y_B^*$. Then y_B^* is a cut vertex in H incident to the cycle and a K_2. Thus one of the neighbors of y_B^* in T corresponds to the cycle and has degree greater than 3 in T. Denote the other neighbor of y_B^* in H by v_1. For v_1 (distance 2 to y_B in H) we may argue as in the previous cases. □

So for every leaf block B we identified its own path (except the end vertices) in T beginning and ending in a vertex of degree at least 3. Let ℓ be the number of leaf blocks in H and so the number of leaves in the tree T. Because of the above results T contains a subgraph consisting of ℓ paths where the sets of inner vertices of the paths are pairwise disjoint. If we replace each path by an edge between its end vertices the set of paths becomes a forest with ℓ edges and at least $\ell + 1$ vertices. Therefore T has at least $\ell + 1$ vertices of degree at least 3. Denote the set of the vertices of degree at least 3 in T by D and the set of leaves in T by $L(T)$.

Now we are ready for the contradiction. By a well-known equality on the number of leaves in a tree we obtain:

$$\ell = |L(T)| = 2 + \sum_{v \in V(T) \setminus L(T)} (d_T(v) - 2)$$
$$\geq 2 + \sum_{v \in D} (d_T(v) - 2) \geq 2 + \ell + 1.$$

Obviously, the inequality gives a contradiction and the proof of the theorem is complete. □

At some places in the paper it is mentioned that the arguments do not work for $d(W) \leq 5$. In fact there are examples showing that $d(W) = 5$ is not sufficient for an extension of a precoloring of an independent set W to a proper list coloring of the whole vertex set. In Fig. 3 such an example is given where w_1 and w_2 are precolored by color a and $L(v) = (a, b, c)$ for all other vertices. Note that this example works also for ordinary colorings and also in that case $d(W) \geq 6$ is the tight bound.

Summarizing the results concerning this topic we obtain: the following corollary.

Corollary. *Let $G = (V, E)$ be a graph with $k = \Delta(G)$ which does not contain K_{k+1} as a subgraph, $W \subseteq V$ an independent subset of the vertex set and $\mathcal{L}$ a list assignment with $|L(v)| = \Delta(G)$ for all $v \in V$.*

Then every precoloring of W extends to a proper $\mathcal{L}$-list coloring of V assuming one of the following

(a) $\Delta(G) \geq 3$ *and* $d(W) \geq 8$ [8,5]
(b) *G is 2-connected,* $\Delta(G) \geq 4$ *and* $d(W) \geq 4$ [15]
(c) *G is 2-connected,* $\Delta(G) = 3$ *and* $d(W) \geq 6$.

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